Problem (4). *Express* $\iiint_E f(x,y,z)dV$ *as an iterated integral in the two different ways below, where* E *is the solid bounded by the surfaces* $y = 4 - x^2 - z^2$ *and* $y = 0$.

(a) $\iiint_E f(x,y,z)dy\,dz\,dx$

(b) $\iiint_E f(x,y,z)dz\,dy\,dx$

Solution

(a) This one is fairly easy. The base in the xz-plane is a circle of radius 2, so we have $-2 \le x \le 2$ and $-\sqrt{4-x^2} \le z \le \sqrt{4-x^2}$. The y bounds are given, so

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{4-x^2-z^2} f(x,y,z)dy\,dz\,dx$$

(b) This one is more difficult. Let's do the systematic approach mentioned in class today: First we find the x bounds, which are the same as before. Now for a fixed x, we have two curves: $y = 0$ and $y = 4 - x^2 - z^2$. Solving for z in terms of y in the second curve, we find $z = \pm\sqrt{4 - x^2 - y}$. Hence our bounds become

$$\int_{-2}^{2}\int_{0}^{4-x^2}\int_{-\sqrt{4-x^2-y}}^{\sqrt{4-x^2-y}} f(x,y,z)dz\,dy\,dx$$

This can be most easily seen by sketching the region in $\mathbb{R}^2$ (remember that here x is a fixed number!) defined by $0 \le y \le 4 - x^2 - z^2$ in the yz-plane; it's bounded by a parabola. Working left to right and bottom to top gives us exactly these bounds.

Remark. Although this doesn't guarantee that the bounds are right, this can be checked by integrating some "easy" function, such as 1. Each integral gives 8π, which is some decent evidence that our answer is correct.

Problem (5). *Evaluate the triple integral* $\iiint_E z dV$ *where* E *is the region in the first octant that lies between the spheres* $x^2 + y^2 + z^2 = 1$ *and* $x^2 + y^2 + z^2 = 4$.

Solution This is best done in spherical coordinates: We have $1 \le \rho \le 2$. Since we're in the first octant, $0 \le \theta, \varphi \le \pi/2$, so we get (recall $z = \rho\cos\varphi$):

$$\int_{0}^{\pi/2}\int_{0}^{\pi/2}\int_{1}^{2} (\rho\cos\varphi)\rho^2\sin\varphi d\rho d\varphi d\theta$$

This iterated integral can be separated into three integrals: The θ integral is equal to $\pi/2$, the ρ integral is equal to $(2^4 - 1^4)/4 = 15/4$, and the φ integral is

$$\int_{0}^{\pi/2}\cos\varphi\sin\varphi d\varphi = \frac{1}{2}\sin^2\varphi\Big|_0^{\pi/2} = \frac{1}{2}$$

Hence, the answer is $15\pi/16$.

Problem (7)**.** *Find the work done by the force field $\vec{F}(x,y) = \vec{i} + (2y+1)\vec{j}$ in moving an object along an arch of the cycloid*

$$\vec{r}(t) = (t - \sin t)\vec{i} + (1 - \cos t)\vec{j}, \quad 0 \le t \le 3\pi$$

Solution The work is the line integral of $\vec{F} \cdot d\vec{r}$. We compute $d\vec{r}$ first:

$$d\vec{r} = \vec{r}'(t)dt = ((1 - \cos t)\vec{i} + \sin t\vec{j})dt$$

Hence,

$$\begin{aligned} \vec{F}(x,y) \cdot r'(t) &= \left(\vec{i} + (2(1 - \cos t) + 1)\vec{j}\right) \cdot \left((1 - \cos t)\vec{i} + \sin t\vec{j}\right) \\ &= (1 - \cos t) + (3 - 2\cos t)\sin t \end{aligned}$$

Thus,

$$W = \int_0^{3\pi} (1 - \cos t) + (3 - 2\cos t)\sin t \, dt$$

Computing the integral is left to the reader.

Problem (9)**.** *Consider the vector field $\vec{F}(x,y) = (7ye^{7x})\vec{i} + (e^{7x} + 2y)\vec{j}$.*

(a) Use a systematic approach to find a potential function for $\vec{F}$. Even if you can do it in your head, instead show work.

(b) Evaluate $\int_C F \cdot d\vec{r}$ where C is parametrized by $\vec{r}(t) = \cos t\vec{i} + t\vec{j}$.

Solution

(a) The systematic approach is to integrate in one variable and differentiate in the other variable, finding the "constant of integration." For no reason whatsoever, let's integrate in y first:

$$f(x,y) = \int e^{7x} + 2y \, dy = ye^{7x} + y^2 + c(x)$$

Now to find c, differentiate in x:

$$7ye^{7x} = \frac{\partial f}{\partial x} = 7ye^{7x} + c'(x)$$

So $c'(x) = 0$ and c is constant. So one potential function is

$$f(x,y) = ye^{7x} + y^2$$

or adding a constant.

(b) The path is irrelevant by the FTC for line integrals. The starting point ($t = 0$) is at $(1,0)$ and the ending point is at $(1, 2\pi)$. Hence,

$$\int_C \vec{F} \cdot d\vec{r} = f(1, 2\pi) - f(1, 0) = 2\pi e^7 + 4\pi^2$$

Problem (11). *Express the volume in the first octant bounded by the plane $x + y + z = 1$ in spherical coordinates.*

Solution Since we're in the first octant, we have $0 \leq \theta, \varphi \leq \pi/2$ (sketch this tetrahedron and convince yourself that we really do need *all* these angles). As far as the radius ρ, we have a lower bound of $\rho = 0$ (at the origin) and the upper bound is expressed by $x + y + z = 1$. Substituting in what x, y, z are in spherical coordinates,

$$\rho \sin\varphi \cos\theta + \rho \sin\varphi \sin\theta + \rho \cos\varphi = 1$$

so our upper bound is

$$\rho = (\sin\varphi \cos\theta + \sin\varphi \sin\theta + \cos\varphi)^{-1}$$

Hence,

$$V = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{(\sin\varphi\cos\theta + \sin\varphi\sin\theta + \cos\varphi)^{-1}} \rho^2 \sin\varphi d\rho d\varphi d\theta$$

Problem (14). *Set up **but do not evaluate** the iterated integral for computing the volume of a region D if D is the right circular cylinder whose base is the disk $r = 2\cos\theta$ (in the xy-plane) and whose top lies in the plane $z = 5 - 2x$.*

Solution Our bounds are already partially given in cylindrical coordinates, so this is the most natural choice. Note z becomes $5 - 2r\cos\theta$. Finally, we need the θ bounds: Note that $r = 2 - \cos\theta$ is a circle centered at the point $(1, 0)$, and passes through the origin at $\theta = 0$ and $\theta = 2\pi$. These are our bounds, so

$$V = \int_0^{2\pi} \int_0^{2-\cos\theta} \int_0^{5-2r\cos\theta} 1\, r\, dz\, dr\, d\theta$$

Problem (16). *Find the volume of the solid region $E = \{(x, y, z) | 0 \leq x \leq z, 1 \leq y \leq 5, y \leq z \leq 5\}$.*

Solution Rectangular coordinates work here. We should integrate in y last, and x before z, since our x bounds are given in terms of z. Hence,

$$V = \int_1^5 \int y^5 \int_0^z 1\, dx\, dz\, dy$$

Computing this integral is left to the reader.

Problem (18). *The density of the half-hemisphere defined by*

$$x^2 + y^2 + z^2 \leq 4, \qquad z \geq 0$$

is equal to the distance above the xy-plane. Find the mass of this object.

Solution This is most natural in spherical coordinates: The upper half-space means $0 \leq \varphi \leq \pi/2$, while $0 \leq \rho \leq 2$ and $0 \leq \theta \leq 2\pi$. Then the density is equal to $z = \rho\cos\varphi$, so

$$m = \int_0^{2\pi} \int_0^{\pi/2} \int_0^2 \rho \cos\varphi\, \rho^2 \sin\varphi d\rho d\varphi d\theta$$

Computing this integral is left to the reader.